A Note on the Galerkin Method

Reza Malek-Madani

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Consider the PDE

$$u_t + L[u] = 0, \tag{1}$$

for $(x, y) \in \Omega$, subject to the boundary conditions

$$u_{|\partial\Omega} = 0, \tag{2}$$

and initial condition

$$u(x, y, 0) = g(x, y) \tag{3}$$

where g is a known function. Let ϕ_n be a set of functions defined in Ω and satisfying (2)-(3). We assume that these functions form a basis for the set of squre-integrable function in Ω, i. e., any function f with the property that

$$\int_\Omega |f|^2 \, dxdy < \infty.$$

Any such function can be expanded in terms of ϕ_n:

$$f = \sum_{n=1}^{\infty} a_n \phi_n. \tag{4}$$

When the functions ϕ_n are orthogonal, i. e.,

$$\int_\Omega \phi_m(x, y)\phi_n(x, y) \, dxdy = 0, \quad \text{if} \qquad m \neq n, \tag{5}$$

then it is easy to determine a_n: Multiply both sides of (4) by ϕ_i, where i is a fixed index, and integrate the result over Ω to get

$$\int_\Omega f(x, y)\phi_i(x, y) \, dxdy = \sum_{n=1}^{\infty} a_n \int_\Omega \phi_n(x, y)\phi_i(x, y) \, dxdy.$$

But, according to (5), the only integral on the right-hand side that survives is $\int_\Omega phi_i^2(x, y) \, dxdy$, hence the above expression reduces to

$$\int_\Omega f(x, y)\phi_i(x, y) \, dxdy = a_i \int_\Omega \phi_i^2(x, y) \, dxdy$$

from which one computes

$$a_i = \frac{\int_\Omega f(x,y)\phi_i(x,y)\,dxdy}{\int_\Omega \phi_i^2(x,y)\,dxdy}. \tag{6}$$

Formula (6) is the familiar formula of Fourier.

Let's use the short-hand notation (f,g) for the integral of the product of f and g, i. e.,

$$(f,g) = \int_\Omega f(x,y)g(x,y)\,dxdy. \tag{7}$$

In terms of (7), the Fourier formula (6) can be written as

$$a_i = \frac{(f,\phi_i)}{(\phi_i,\phi_i)}, \qquad i = 1,2,... \tag{8}$$

Now let's go back to our original initial-boundary value problem. The Galerkin method suggests seeking a solution of (1), (2), (3) in the form

$$u(x,y,t) = \sum a_n(t)\phi_n(x,y) \tag{9}$$

for a basis ϕ_n which satisfies the boundary condition in (2). Since the ansatz (2) is to be a solution of the PDE, it must satisfy that equation, so we substitute it into (1) to get

$$\sum_{n=1}^{\infty} a_n'(t)\phi_n(x,y) + L[\sum_{n=1} a_n(t)\phi_n(x,y)] = 0.$$

We next multiply both sides of this equation by ϕ_i, for a fixed i and integrate over Ω. Using the notation in (7) we have

$$\sum_{n=1}^{\infty} a_n'(t)(\phi_n,\phi_i) + (L[\sum_{n=1}^{\infty} a_n\phi_n],\phi_i] = 0, \qquad i = 1,2,... \tag{10}$$

Also, after evaluating (9) at $t=0$ and using the information in (3), we have

$$g(x,y) = \sum_{n=1}^{\infty} a_n(0)\phi_n(x,y).$$

We multiply this equation by ϕ_i and integrate over Ω to get

$$(g,\phi_i) = \sum_{n=1}^{\infty} a_n(0)(\phi_n,\phi_i), \qquad i = 1,2,... \tag{11}$$

Note that because g and ϕ are known functions, all of the integrals of the form (g,ϕ_i) and (ϕ_n,ϕ_i) can be computed, often rather easily and inexpensively. This way, our initial-boundary value problem (1), (2), (3) is redeuced to a system of inifinitely many ordinary differential equations given by (10) with initial data given in (11). A large body of the mathematics of the latter half of the twentieth century has been dedicated to understanding under what conditions our initial-boundary value problem is equivalent to the inifinite system of ODEs described above.

One of the most desirable aspects of the Galerkin method is that it applies to nonlinear problems as well as to linear problems. Another desiable property is the ease with which one can implement this technique on a computer, especially when the geometry of the domain is simple. The implementation is particularly simple when the basis functions are orthogonal. The following examples illustrate these points.

1 Example 1: Heat Equation

Let's consider the case when Ω is a unit square and

$$L[u] = -\Delta u.$$

Then (1) becomes

$$u_t - \Delta u = 0. \tag{12}$$

Let

$$\phi_{mn} = \sin n\pi x \sin n\pi y. \tag{13}$$

(why do we introduce ϕ_{mn} in place of ϕ_n?) Note that ϕ_{mn} already satisfies the zero boundary condition on the boundary of the unit square. The ansatz in (9) takes the form

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}(t) \sin n\pi x \sin m\pi y. \tag{14}$$

It is easy to show that ϕ_{mn} are orthogonal, i.e.,

$$\int_0^1 \int_0^1 \sin n\pi x \sin m\pi y \sin i\pi x \sin j\pi y \, dxdy = 0$$

if $i \neq n$ and $j \neq m$. Substituting (14) into (12) yields

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a'_{mn}(t)\phi_{mn} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\Delta\phi_{mn} = 0.$$

Multiply the above equation by ϕ_{ij} and integrate over Ω to get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mnij}a'_{mn}(t) - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} d_{mnij}a_{mn}(t) = 0, \tag{15}$$

where

$$c_{mnij} = \int_0^1 \int_0^1 \sin m\pi x \sin n\pi y \sin i\pi x \sin j\pi y \, dxdy \qquad \text{and}$$

$$d_{mnij} = \int_0^1 \int_0^1 \Delta(\sin m\pi x \sin n\pi y) \sin i\pi x \sin j\pi y \, dxdy.$$

It is easy to compute the above integrals using the orthogonality of the sine functions. In fact $c_{mnij} = \frac{1}{4}\delta_{mi}\delta_{nj}$ and $d_{mnij} = -\frac{m^2+n^2}{4}\delta_{mi}\delta_{nj}$.

Let

$$u(x, y, 0) = g(x, y) \tag{16}$$

define the initial state of u. The differential equations in (15) are supplemented with the initial data as described in (11). In practice, we replace ∞ by M in (16) and (11) and solve the resulting M^2 ODEs numerically to get an approximate solution of our problem. The following program in Mathematica will do the job.

2 Example 2: Reaction Diffusion Equation

Consider the PDE

$$u_t = u(1 - u) + \Delta u \tag{17}$$

in the unit square. We choose the same basis function and ansatz as in Example 1 (see (14). Now substitute (14) into (17) to get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a'_{mn}(t)\phi_{mn} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}(t)\phi_{mn} - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{mn}(t)a_{kl}(t)\phi_{mn}\phi_{kl} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\Delta\phi_{mn}.$$

The quadruple sum in the above expression is due to the u^2 in (17). As before we multiply this equation by ϕ_{ij} and integrate over Ω to get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{mnij}a'_{mn}(t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mnij}a_{mn}(t) - \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} d_{mnklij}a_{mn}(t)a_{kl}(t) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e_{mnij}a_{mn}(t) \tag{18}$$

where the coefficients b, c, d and e involve the various integrals of ϕ_{mn}. The above system is considerably different from the one we get from the heat equation in that the latter is a **nonlinear** system. This fact presents a challenge to any computing system when we replace ∞ with M^2 since we are often interested in the large time (steady-state) behavior of this problem so the M^2 ODEs need to be integrated over a long time interval. The following Mathematica program shows how one handles this problem when M is in the range of 5 to 10.